HEAT CONDUCTION IN A MULTILAYER COMPOSITE WEDGE

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A new method for analytically solving a problem of steady-state heat conduction for multilayer composite wedge-shaped bodies is suggested based on a generalization of the integral Mellin transform.

We consider a plane steady-state heat conduction problem for a periodically nonuniform composite body having the form of a sector with an aperture angle $2\theta_0$ and consisting of individual layers of the same thickness ε (see Fig. 1). If we assume that the thermal conductivity coefficients λ_{rr}, $\lambda_{\theta\theta}$ within the limits of each layer are known functions of the variable r, then the heat flux density vector is of the form

$$q = (q_r,\ q_\theta,\ q_z), \tag{1}$$

$$q_r = -\lambda_{rr}^{(\varepsilon)} \frac{\partial T}{\partial r}, \quad q_\theta = -\lambda_{\theta\theta}^{(\varepsilon)} \frac{1}{r} \frac{\partial T}{\partial \theta}, \quad q_z = 0,$$

where $\lambda_{rr}^{(\varepsilon)} = \lambda_{rr}(\rho)$, $\lambda_{\theta\theta}^{(\varepsilon)} = \lambda_{\theta\theta}(\rho)$ are singly periodic functions of the variable $\rho = r/\varepsilon$, which may be called "rapid." On the basis of (1) we obtain the heat conduction equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r\lambda_{rr}(\rho) \frac{\partial T}{\partial r} \right) + \lambda_{\theta\theta}(\rho) \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = 0, \tag{2}$$

whose solution is determined in the unbounded region $0<r<\infty$, $|\theta| < \theta_0$ under the prescribed conditions at the wedge boundary ($\theta = \pm\theta_0$).

Since with the constant thermal conductivity coefficients λ_{rr} and $\lambda_{\theta\theta}$ the solution to the boundary-value problem of Eq. (2) can be obtained analytically using the integral Mellin transform [1], we are interested in an elaboration of the generalized Mellin transform by means of which it is possible to solve the boundary-value problem of Eq. (2) with the periodic coefficients $\lambda_{rr}(\rho)$ and $\lambda_{\theta\theta}(\rho)$. For this purpose we make use of a general scheme of the method [2] and consider the equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left(rA(\rho) \frac{\partial u}{\partial r} \right) = B(\rho) \frac{1}{r^2} \frac{\partial u}{\partial t} \tag{3}$$

with the singly periodic coefficients $A(\rho)$ and $B(\rho)$. The solution of (3) should be sought within the region $0<r<\infty$, $t>0$ under the initial condition

$$u|_{t=0} = f(r). \tag{4}$$

Having applied to (3) the Laplace transform with respect to the variable t, we get the ordinary differential equation

$$\frac{1}{r} \frac{d}{dr} \left(rA(\rho) \frac{d\bar{u}}{dr} \right) - p\frac{B(\rho)}{r^2} \bar{u} = -B(\rho) \frac{1}{r^2} f(r), \tag{5}$$

whose solution can be written as

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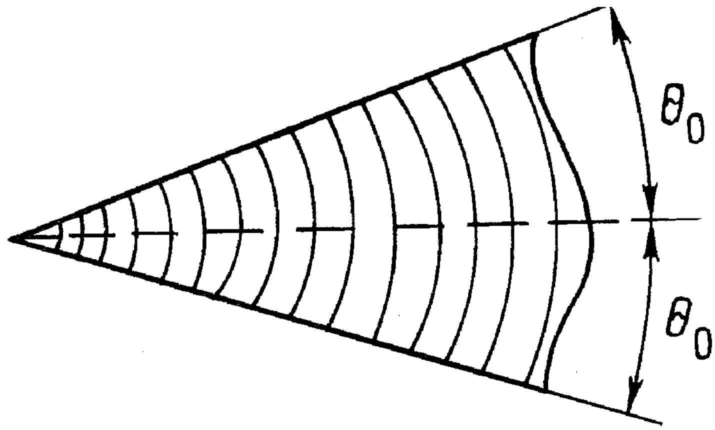

Fig. 1.

$$\bar{u}(r,\ p)=-\frac{1}{\omega(p)}\int\limits_0^\infty G(r,\ \xi,\ p)\,B\left(\frac{\xi}{\varepsilon}\right)f(\xi)\frac{d\xi}{\xi}. \tag{6}$$

Here

$$G(r,\ \xi,\ p)=\begin{cases}\bar{u}_1(r,\ p)\bar{u}_2(\xi,\ p), & r\leqslant\xi,\\ \bar{u}_1(\xi,\ p)\bar{u}_2(r,\ p), & r\geqslant\xi,\end{cases}$$

$$\omega(p)=rA(\rho)\,W(\bar{u}_1,\ \bar{u}_2),$$

and $W(\bar{u}_1, \bar{u}_2)$ is the Wronskian of the linearly independent solutions of $\bar{u}_1$, $\bar{u}_2$ in the homogeneous equation (5).

Passing on to the construction of the independent solutions of $\bar{u}_1$, $\bar{u}_2$ for the homogeneous equation

$$\frac{1}{r}\frac{d}{dr}\left(rA(\rho)\frac{d\bar{u}}{dr}\right)-p\frac{B(\rho)}{r^2}\bar{u}=0, \tag{7}$$

we will seek them in series form with respect to the small parameter ε [3, 4]

$$\bar{u}=u_0(r)+\varepsilon u_1(r,\ \rho)+\varepsilon^2u_2(r,\ \rho)+\ldots \tag{8}$$

Substituting (8) into (7), it can be shown that the functions $u_1(r, \rho)$ and $u_2(r, \rho)$ are defined in the following manner:

$$u_1(r,\ \rho)=N_1(\rho)u_0'(r),\quad u_2(r,\ \rho)=N_2^{(1)}(\rho)u_0''(r)+N_2^{(2)}(\rho)\frac{1}{r}(ru_0')', \tag{9}$$

where $u_0(r)$ is the solution of the averaged equation

$$u_0''(r)+\frac{1}{r}u_0'(r)-p\frac{\varkappa^2}{r^2}u_0(r)=0\quad(\varkappa^2=\langle B\rangle\ \langle A^{-1}\rangle). \tag{10}$$

The local functions $N_1(\rho)$, $N_2^{(1)}(\rho)$, $N_2^{(2)}(\rho)$ are the singly periodic solutions to the equations $(0\leq\rho\leq1)$

$$\frac{d}{d\rho}\left(A(\rho)\frac{dN_1}{d\rho}+A(\rho)\right)=0, \tag{11}$$

$$\frac{dN_2^{(1)}}{d\rho}=-N_1(\rho), \tag{12}$$

$$\frac{d}{d\rho}\left[A(\rho)\frac{dN_2^{(2)}}{d\rho}\right]+C_1\left(1-\frac{B(\rho)}{\langle B\rangle}\right)=0. \tag{13}$$

Here

$$C_1 = A(\rho)\frac{dN_1}{d\rho} + A(\rho) = \frac{1}{\langle A^{-1}\rangle}, \quad \langle B \rangle = \int_0^1 B(\rho)\,d\rho$$

and the solutions of (11) and (12) satisfy the conditions

$$\langle N_1 \rangle = 0, \quad \langle N_2^{(1)} \rangle = 0\,. \tag{14}$$

If we now choose the independent solutions for the averaged equation (10) in the form

$$u_{01} = r^{\varkappa\sqrt{p}}, \quad u_{02} = r^{-\varkappa\sqrt{p}}, \tag{15}$$

then, in accordance with (8), (9), we obtain the following expressions for the linearly independent solutions of the homogeneous equation (7):

$$\begin{gathered}\bar{u}_1 = r^{\varkappa\sqrt{p}} + \varepsilon\varkappa\sqrt{p}N_1(\rho)\,r^{\varkappa\sqrt{p}-1} + \varepsilon^2\,[(\varkappa^2 p - \varkappa\sqrt{p})\,N_2^{(1)}(\rho) + \\ + \varkappa^2 p N_2^{(2)}(\rho)]\,r^{\varkappa\sqrt{p}-2} + \ldots, \\ \bar{u}_2 = r^{-\varkappa\sqrt{p}} - \varepsilon\varkappa\sqrt{p}N_1(\rho)\,r^{-\varkappa\sqrt{p}-1} + \\ + \varepsilon^2\,[(\varkappa^2 p + \varkappa\sqrt{p})\,N_2^{(1)}(\rho) + \varkappa^2 p N_2^{(2)}(\rho)]\,e^{-\varkappa\sqrt{p}-2} + \ldots\end{gathered} \tag{16}$$

Using (16) and taking into account (14), we get for the Wronskian $W(\bar{u}_1, \bar{u}_2)$:

$$W(\bar{u}_1, \bar{u}_2) = -\frac{2\varkappa\sqrt{p}C_1}{rA(\rho)}. \tag{17}$$

Now, with allowance for (6), (17) we find

$$\bar{u}(r, p) = \frac{1}{2\varkappa C_1}\int_0^\infty \frac{G(r, \xi, p)}{\sqrt{p}}\,B\left(\frac{\xi}{\varepsilon}\right)f(\xi)\,\frac{d\xi}{\xi}, \tag{18}$$

and passing in (18) to the original, we obtain

$$u(r, t) = \frac{1}{2\varkappa C_1}\int_0^\infty J(r, \xi, t)\,B\left(\frac{\xi}{\varepsilon}\right)f(\xi)\,\frac{d\xi}{\xi}, \tag{19}$$

where the notation

$$J(r, \xi, t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G(r, \xi, p)\,e^{pt}/\sqrt{p}\,dp. \tag{20}$$

is introduced.

The integrand in (20) has one branching point $p = 0$. Performing a cut along the real axis from $-\infty$ up to $p = 0$, it is possible to express $J(r, \xi, t)$ through integrals over the cut edges of the complex variable p. Using further the values of $\bar{u}_1$ and $\bar{u}_2$ according to (16), we find, after some transformations, the following expression for $J(r, \xi, t)$:

$$\begin{gathered}J(r, \xi, t) = \frac{1}{\pi i\varkappa}\int_{-i\infty}^{i\infty} r^p m(r, p)\,\xi^{-p} m(\xi, -p)\,e^{p^2 t/\varkappa^2}dp = \\ = \frac{1}{\pi i\varkappa}\int_{-i\infty}^{i\infty} r^{-p} m(r, -p)\,\xi^p m(\xi, p)\,e^{p^2 t/\varkappa^2}dp.\end{gathered} \tag{21}$$

Here

$$\begin{gathered}m(r, p) = 1 + \varepsilon p N_1\left(\frac{r}{\varepsilon}\right)\frac{1}{r} + \\ + \varepsilon^2\left[(p^2 - p)\,N_2^{(1)}\left(\frac{r}{\varepsilon}\right) + p^2 N_2^{(2)}\left(\frac{r}{\varepsilon}\right)\right]\frac{1}{r^2} + \ldots\end{gathered} \tag{22}$$

Now the unknown expansion is obtained on substituting (21) into (19), if we assume that in the last formula, in accordance with (4), t = 0

$$f(r)=\frac{1}{\langle B\rangle}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} r^{-p}m(r,\ -p)\left(\int_{0}^{\infty}\xi^{p-1}m(\xi,\ p)f(\xi)B\left(\frac{\xi}{\varepsilon}\right)d\xi\right)dp. \tag{23}$$

Thus, using Eq. (23), we obtain the formulas which determine the generalization of the integral Mellin transform as

$$f(r)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} r^{-p}m(r,\ -p)\tilde{f}(p)\,dp, \tag{24}$$

$$\tilde{f}(p)=\frac{1}{\langle B\rangle}\int_{0}^{\infty}\xi^{p-1}m(\xi,\ p)f(\xi)B\left(\frac{\xi}{\varepsilon}\right)d\xi. \tag{25}$$

Note that the function $r^{-p}m(r, -p)$ is the approximate solution to the equation

$$\frac{1}{r}\frac{d}{dr}\left[rA(\rho)\frac{d}{dr}(r^{-p}m(r,\ -p))\right]-\frac{p^2B(\rho)}{\varkappa^2r^2}(r^{-p}m(r,\ -p))=0, \tag{26}$$

which can be verified by differentiation of the expression $r^{-p}m(r, -p)$ with account of the formula (22).

On the basis of the obtained generalized integral Mellin transform of (24), (25), (22), we find the solution to the heat conduction equation of (2) within the region $0<r<\infty$, $|\theta|<\theta_0$ under the following boundary conditions (see Fig. 1):

$$T(r,\ \theta_0)=f_1(r),\quad T(r,\ -\theta_0)=f_2(r). \tag{27}$$

Assuming that in the representations of (22), (24), (25), $A(\rho)=\lambda_{rr}(\rho)$, $B(\rho)=\lambda_{\theta\theta}(\rho)$, we will seek the solution (2) in the form

$$T(r,\ \theta)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} r^{-p}m(r,\ -p)\tilde{T}(p,\ \theta)\,dp. \tag{28}$$

Then, with allowance for (26) for the function $\tilde{T}(p,\theta)$, we obtain the ordinary differential equation

$$\frac{d^2\tilde{T}(p,\ \theta)}{d\theta^2}+\frac{p^2}{\varkappa^2}\tilde{T}(p,\ \theta)=0,$$

from which

$$\tilde{T}(p,\ \theta)=C(p)\cos\left(\frac{p\theta}{\varkappa}\right)+D(p)\sin\left(\frac{p\theta}{\varkappa}\right). \tag{29}$$

Substituting (29) into (28) and satisfying the boundary conditions of (27), we have the equalities

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} r^{-p}m(r,\ -p)\left[C(p)\cos\left(\frac{p\theta_0}{\varkappa}\right)+D(p)\sin\left(\frac{p\theta_0}{\varkappa}\right)\right]dp=f_1(r), \tag{30}$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} r^{-p}m(r,\ -p)\left[C(p)\cos\left(\frac{p\theta_0}{\varkappa}\right)-D(p)\sin\left(\frac{p\theta_0}{\varkappa}\right)\right]dp=f_2(r). \tag{31}$$

Applying the inverse transform of (25) to expressions (30) and (31), we arrive at a set of equations with respect to the functions C(p), D(p):

$$C(p)\cos\left(\frac{p\theta_0}{\varkappa}\right)+D(p)\sin\left(\frac{p\theta_0}{\varkappa}\right)=$$

$$=\frac{1}{\langle\lambda_{\theta\theta}\rangle}\int_{0}^{\infty}\xi^{p-1}m(\xi,\ p)f_1(\xi)\lambda_{\theta\theta}\left(\frac{\xi}{\varepsilon}\right)d\xi, \tag{32}$$

$$C(p)\cos\left(\frac{p\theta_0}{\varkappa}\right) - D(p)\sin\left(\frac{p\theta_0}{\varkappa}\right) =$$

$$= \frac{1}{\langle \lambda_{\theta\theta} \rangle} \int_0^\infty \xi^{p-1} m(\xi,\ p) f_2(\xi) \lambda_{\theta\theta}\left(\frac{\xi}{\varepsilon}\right) d\xi.$$

The solution of Eq. (32) yields

$$\begin{aligned} C(p) &= (1/2)[g_1(p) + g_2(p)] \cos^{-1}\left(\frac{p\theta_0}{\varkappa}\right), \\ D(p) &= (1/2)[g_1(p) - g_2(p)] \sin^{-1}\left(\frac{p\theta_0}{\varkappa}\right). \end{aligned} \tag{33}$$

Here

$$g_k(p) = \frac{1}{\langle \lambda_{\theta\theta} \rangle} \int_0^\infty \xi^{p-1} m(\xi,\ p) f_k(\xi) \lambda_{\theta\theta}\left(\frac{\xi}{\varepsilon}\right) d\xi \quad (k = 1,\ 2).$$

Thus, the formulas (28), (29), and (33) are the analytical solution of the heat conduction problem (2), (27) in a multilayer composite wedge. The nonuniformity character inside each of the layers is simulated by the functions $\lambda_{rr}(\rho)$ and $\lambda_{\theta\theta}(\rho)$. These functions, in particular, may have the nature of piecewise-constant functions, taking different values in regions of various components of the composite material. The behavior of these functions is manifested in the form of the solution of the local problems (11)-(14) and, through the local functions $N_1(\rho)$, $N_2^{(1)}(\rho)$, $N_2^{(2)}(\rho)$, has an effect on the function m(r, p) (22), which enters into the solution formulas. It should be pointed out as well that in the limiting special case of uniform material (at constant λ_{rr} and $\lambda_{\theta\theta}$), from the local problems of (11)-(14) it follows that all the local functions are equal to zero and the transformation of (24) and (25) with account of (22) is reduced to the integral Mellin transform.

NOTATION

T, temperature; λ_{rr}, $\lambda_{\theta\theta}$, thermal conductivity coefficients; ε, thickness of composite material layers ($\varepsilon \ll 1$); $N_1(\rho)$, $N_2^{(1)}(\rho)$, $N_2^{(2)}(\rho)$, auxiliary local functions from the "rapid" variable $\rho = r/\varepsilon$; m(r, p), auxiliary function entering the core of the generalized integral Mellin transform; θ_0, half of the wedge aperture angle.

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